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Curvature and Tangency Handles for Control of Convex Cubic Shapes

G. Figueroa, M. Paluszny, and F. Tovar

Abstract. We consider the problem of modelling a plane convex shape with a closed component of an algebraic non-singular cubic. All nine degrees of freedom are interpreted as visual handles, namely: tangency to three prescribed lines at three given points and the curvatures at these points.

§1. Introduction

Algebraic curves, beyond conics, where introduced in CAGD by Sederberg [6] in 1984. Paluszny and Patterson [1–3] studied splines constructed with segments of cubic algebraic curves, called A-splines. Tovar, Paluszny and Patterson [4] looked at A-splines constructed with segments of singular algebraic cubics, which are just rational cubics, with new, geometrically more meaningful, control handles for their shape. Non-singular cubics are classically well-known objects Salmon [5]. Projectively they could be of two types: two or one circuit cubics. A two circuit cubic consists of two pieces, one of which can be realized affinely as a convex closed curve, called an oval.

The goal of this paper is to study the feasibility of using ovals to model plane convex shapes. We remark that ovals are not splines, they are C^∞ curves. Therefore, the main advantage of using ovals for modelling of convex shapes is that no stitching of segments is required, which would be the case for splines. In fact, a convex shape represented by a cubic oval doesn't have any joints at all. It seems natural to place the study of cubic ovals within the context of A-splines because they are connected components of cubic algebraic curves. Moreover, the techniques to study the shape handles for their control are similar to those used for A-splines as mentioned above. In particular, we want to control the shape of an oval through its control triangle, contact interpolation, and curvatures at three prescribed points, see Figures 1–4.

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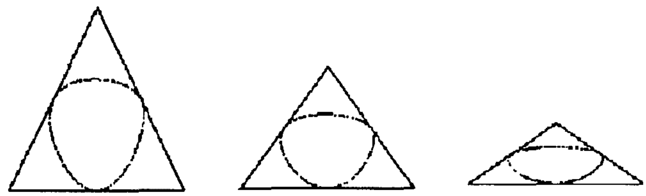

Fig. 1. Shape control by moving vertices of the control triangle.

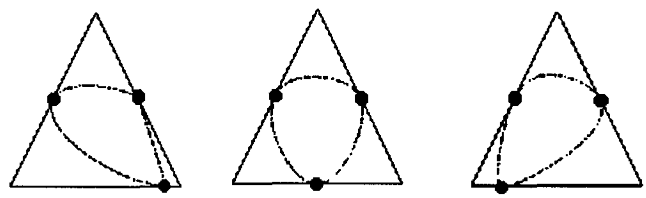

Fig. 2. Shape control by moving one of the tangency points.

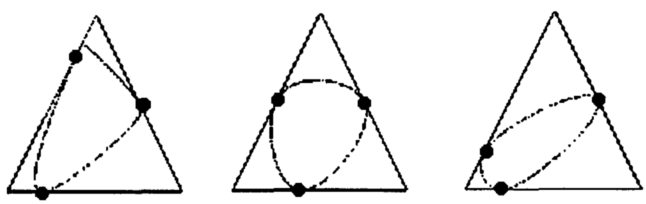

Fig. 3. Shape control by moving two of the tangency points.

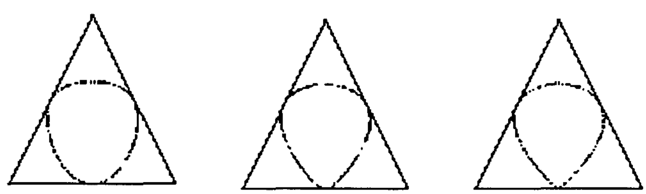

Fig. 4. Sharpening the curvature at one contact point.

§2. Barycentric Coordinates and Curvature at the Endpoints

The general algebraic cubic in cartesian coordinates x, y is given by

$$F(x,y) = a_{30}x^3 + a_{03}y^3 + a_{21}x^2y + a_{12}xy^2 + a_{20}x^2 + a_{02}y^2 + a_{11}xy + a_{10}x + a_{01}y + a_{00}.$$

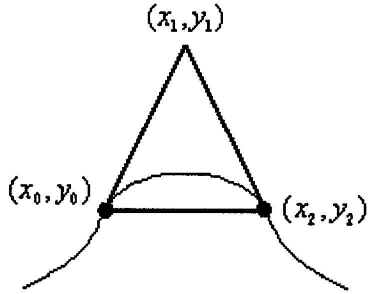

Fig. 5. Cubic with two prescribed tangencies.

If the cubic interpolates two points (x_0, y_0) and (x_2, y_2), and it is tangent to two lines joining these points to a third point (x_1, y_1) as shown in Figure 5, then its expression in terms of the barycentric coordinates (S, T, U) with respect to the triangle of vertices (x_0, y_0), (x_1, y_1) and (x_2, y_2) reduces to

$$F(S,T,U) = aS^2U + bSU^2 - cST^2 - dT^2U + eSTU + fT^3 \tag{1}$$

where a, b, c, d, e and f are arbitrary real coefficients. This was observed by Sederberg [6]. In Paluszny and Patterson [2] it was shown that the curvatures $\bar{k}_s$ and $\bar{k}_u$ of (1) at (x_0, y_0) and (x_2, y_2) are given by

$$\bar{k}_u = \frac{c}{a} \frac{\Delta}{(\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2})^3},$$

$$\bar{k}_s = \frac{d}{b} \frac{\Delta}{(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2})^3},$$

where Δ is the area of the triangle with vertices (x_0, y_0), (x_1, y_1) and (x_2, y_2).

§3. Three Prescribed Contacts

We now focus on the family of cubics with three prescribed contacts. Consider a triangle of vertices P_0, P_1 and P_2, and let (s, t, u) be the corresponding barycentric coordinates. Choose one point on each side, as shown in Figure 6.

The barycentric coordinates of each Q_i are $Q_0 = Q_0(0, t_0, u_0)$, $Q_1 = Q_1(s_1, 0, u_1)$, and $Q_2 = Q_2(s_2, t_2, 0)$. The general equation of a cubic passing through these points and tangent (or singular) at them to the sides of the

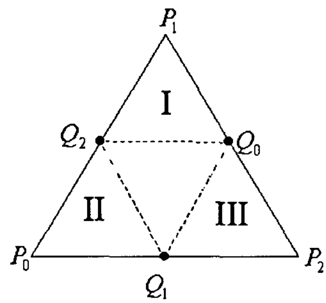

Fig. 6. Barycentric coodinates of side points.

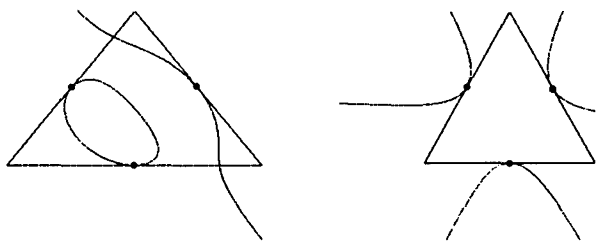

Fig. 7. Wrong contacts.

triangle is

$$
\begin{aligned}
G(s,t,u) &= a_{300}s^3 + a_{030}t^3 + a_{003}u^3 + \frac{(s_2^3 a_{300} - 2a_{030}t_2^3)st^2}{s_2 t_2^2} \\
&+ \quad \frac{(a_{030}t_2^3 - 2s_2^3 a_{300})s^2 t}{s_2^2 t_2} + \frac{(t_0^3 a_{030} - 2a_{003}u_0^3)tu^2}{t_0 u_0^2} \\
&+ \quad \frac{(a_{003}u_0^3 - 2t_0^3 a_{030})t^2 u}{t_0^2 u_0} + \frac{(s_1^3 a_{300} - 2a_{003}u_1^3)su^2}{s_1 u_1^2} \\
&+ \quad \frac{(a_{003}u_1^3 - 2s_1^3 a_{300})s^2 u}{s_1^2 u_1} + a_{111}stu.
\end{aligned}
$$

Note that there are four free homogeneous parameters. These have to be used for two purposes:

- to guarantee that the cubic is actually a two circuit cubic and that the interpolation occurs at points of the oval,
- to express the free parameters in terms of the curvatures at the interpolation points, and find interval ranges for their meaningful modification. The first point is crucial because we need to preclude situations in which the contacts occur at points off the oval, as illustrated in Figure 7.

§4. Curvatures at the Contact Points

We need to control the curvatures at the interpolation points. The first step will be to find formulas for the curvatures in terms of the free parameters a_{300}, a_{030}, a_{003} and a_{111} of the cubic, as expressed in the barycentric coordinates (s,t,u). In the next section we will produce inversion formulas for the a_{ijk} in terms of the curvatures, which will allow us to find oval shapes with prescribed curvatures at the prescribed contacts. It will be convenient to express the curvatures $\bar{k}_s$ and $\bar{k}_u$ at Q_2 and Q_0 respectively, in terms of the barycentric coordinates s,t,u with respect to P_0, P_1 and P_2. As remarked in the previous section

$$\bar{k}_u = \frac{c_I}{a_I}\frac{\Delta_I}{(\overline{P_1Q_0})^3},$$

where Δ_I is the area of triangle I, $\overline{P_1Q_0}$ denotes the distance between P_1 and Q_0, and a_I, b_I, c_I and d_I, are the coefficients of the cubic in the barycentric coordinates with respect to Q_2, P_1 and Q_0, compare with (1). And similarly

$$\bar{k}_s = \frac{d_I}{b_I}\frac{\Delta_I}{(\overline{P_1Q_2})^3}.$$

In fact if a,b,c,d,e,f are the coefficients of the cubic in the barycentric coordinates with respect to the triangle $P_0P_1P_2$ as expressed in (1), then the coefficients a_I, b_I, c_I and d_I of this cubic with respect to the triangle I of vertices Q_2, P_1 and Q_0 are obtained by a linear change of variables:

$$a_I = \frac{u_0u_1(\alpha^2+\beta^2)a_{003} - t_0^2s_1^2(2a_{300}u_0s_2^2s_1 + 2t_0a_{030}t_2^2u_1 - a_{111}s_2t_2u_0u_1)}{s_1^2u_1t_0^2},$$

$$b_I = \frac{s_2s_1(\alpha^2+\beta^2)a_{300} - t_2^2u_1^2(2a_{030}t_0^2s_1t_2 + 2s_2u_0^2a_{003}u_1 - a_{111}s_2t_0u_0s_1)}{t_2^2s_1u_1^2},$$

where $\alpha = u_1s_2t_0$, $\beta = u_0t_2s_1$, and

$$c_I = -\frac{a_{030}t_2^3 + s_2^3a_{300}}{t_2^2},$$

$$d_I = -\frac{a_{003}u_0^3 + a_{030}t_0^3}{t_0^2}.$$

Using the relationship between the areas of the triangles $P_0P_1P_2$ and $Q_2P_1Q_0$ as illustrated in Figure 6, it follows that $\bar{k}_u$ and $\bar{k}_s$ can be expressed directly in terms of the geometry of the triangle $P_0P_1P_2$:

$$\bar{k}_u = \frac{4c_I}{a_I}\frac{u_0}{s_2^2}\frac{\Delta}{(\overline{P_0P_1})^3}, \tag{2}$$

$$\bar{k}_s = \frac{4d_I}{b_I}\frac{s_2}{u_0^2}\frac{\Delta}{(\overline{P_1P_2})^3}, \tag{3}$$

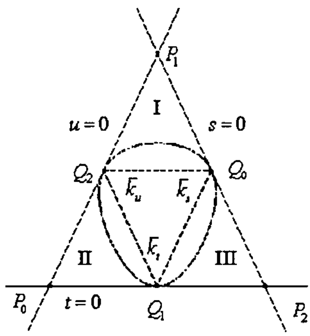

Fig. 8. Curvatures at the contact points.

where Δ is the area of the triangle $P_0P_1P_2$. To compute the curvature $\bar{k}_t$ at Q_1 with barycentric coordinates $(s_1, 0, u_1)$, we consider triangle II to get

$$\bar{k}_t = \frac{4d_{II}}{b_{II}} \frac{t_2}{u_1^2} \frac{\Delta}{(\overline{P_0P_2})^3}. \tag{4}$$

It is easy to show that using triangle III instead, we obtain the same expression for $\bar{k}_t$.

Figure 8 illustrates the relationships between the coordinates of contact points Q_0, Q_1 and Q_2 and curvatures k_s, k_t and k_u.

§5. Inversion Formulas and Shape Control

Equations (2)–(4) can be rewritten as

$$k_u s_2^2 a_I - u_0 c_I = 0$$

$$k_s u_0^2 b_I - s_0 d_I = 0$$

$$k_t u_1^2 a_{II} - t_2 c_{II} = 0,$$

where the a, b, c, d coefficients are given in terms of a_{300}, a_{030}, a_{003}, and a_{111} and k_s, k_t and k_u are proportional to the curvatures $\bar{k}_s$, $\bar{k}_t$ and $\bar{k}_u$, i.e.

$$k_u = \bar{k}_u \frac{\overline{P_0P_1}^3}{\Delta}, \qquad k_t = \bar{k}_t \frac{\overline{P_0P_2}^3}{\Delta}, \qquad k_s = \bar{k}_s \frac{\overline{P_1P_2}^3}{\Delta}.$$

The formulas for the a_{ijk} in terms of k_s, k_t and k_u are

$$\begin{aligned} a_{300} = u_1^2 t_2^2 (k_t k_u (\alpha+\beta)^3 (s_1 s_2 - t_0 u_0 (\alpha+\beta) k_s) \\ + u_1 t_2 (t_0^3 u_0^3 k_s - s_2^3 t_0^3 k_u - s_1^3 u_0^3 k_t)), \end{aligned}$$

$$\begin{aligned} a_{030} = u_0^2 s_2^2 (k_s k_u (\alpha+\beta)^3 (t_0 t_2 - s_1 u_1 (\alpha+\beta) k_t) \\ + s_2 u_0 (u_1^3 s_1^3 k_t - u_1^3 t_0^3 k_s - s_1^3 t_2^3 k_u)), \end{aligned}$$

$$\begin{aligned} a_{003} = t_0^2 s_1^2 (k_s k_t (\alpha+\beta)^3 (u_0 u_1 - t_2 s_2 (\alpha+\beta) k_u) \\ + s_1 t_0 (s_2^3 t_2^3 k_u - t_2^3 u_0^3 k_s - s_2^3 u_1^3 k_t)), \end{aligned} \tag{5}$$

$$\begin{aligned} a_{111} = (\alpha+\beta)^3 (k_s k_t k_u (\alpha+\beta)((\alpha-\beta)^2 - 2\alpha\beta) \\ + 2 k_s k_t u_0^2 t_0 s_1 u_1^2 + 2 k_s k_u u_0 t_0 t_2^2 s_2 + 2 t_2 s_2^2 s_1^2 u_1^2 k_t k_u) \\ - 3(\alpha^2+\beta^2)(t_0 t_2^2 s_2^2 s_1 k_u + u_0^2 t_0^2 t_2 u_1 k_s + u_0 s_2 s_1^2 u_1^2 k_t) \\ + (\alpha+\beta)((\alpha-\beta)^2 + \alpha\beta). \end{aligned}$$

As established above, see Figure 7, further constraints are required for the contacts to occur at points of the oval, and for the latter to be contained inside the triangle $P_0P_1P_2$. Given the points Q_0, Q_1 and Q_2, (2) determines a cubic which is tangent to the sides of triangle $P_0P_1P_2$ at these points. To guarantee that the points Q_0, Q_1 and Q_2 lie on the oval it is sufficient that the coefficients a_{300}, a_{030} and a_{003} have the same sign. Indeed, the first condition implies that the third intersection of the cubic with each of the lines $s = 0$, $t = 0$ and $u = 0$ occurs outside the triangle. Since each contact Q_i accounts for two intersections, if the cubic at every Q_i bends inwards the triangle $P_0P_1P_2$, then the contacts actually occur at points of the oval of the cubic which then has to lie inside the triangle. The positivity of k_s, k_t and k_u guarantees that the curve bends towards the interior of the triangle, see [3].

Given the contacts at $Q_0(0, t_0, u_0)$, $Q_1(s_1, 0, u_1)$ and $Q_2(s_2, t_2, 0)$, see Figure 6, for a_{003}, a_{030} and a_{003} to have the same sign, it is enough to take

$$\begin{aligned} k_s^0 &= \frac{s_2 s_1}{t_0 u_0 (\alpha+\beta)}, \\ k_t^0 &= \frac{t_0 t_2}{u_1 s_1 (\alpha+\beta)}, \\ k_u^0 &= \frac{u_0 u_1}{s_2 t_2 (\alpha+\beta)}. \end{aligned} \tag{6}$$

Note that given an oval with three prescribed curvatures, when we fix two of them the equations for the a_{ijk} in terms of the third are linear. So, it is easy to find an interval range for the modification of the third curvature, while keeping the coefficients a_{300}, a_{030} and a_{003} of the same sign.

§6. Conclusion

A convex shape can be modelled with a cubic oval controlled by a triangle that contains it. By moving the vertices, we drag the oval along (see Figure 1). Moreover, given any triangle and three points, one on each of its sides, we can produce a convex shape that contacts the triangle at these points. This convex shape is our cubic oval. Furthermore, we can prescribe the curvatures at these three points, within precisely defined conditions. If we fix two curvatures, the third curvature can be modified within an interval which can be computed by

solving a system of linear inequalities. In fact given the vertices of the control triangle, the contact points at its sides and the three initial curvatures k_s^0, k_t^0 and k_u^0 as given by (6), it is always possible to modify any of them keeping the other two fixed.

This entails solving the linear inequalities $a_{300} > 0$, $a_{030} > 0$ and $a_{003} > 0$, or $a_{300} < 0$, $a_{030} < 0$ and $a_{003} < 0$ given by the linear system (5).

References

1. Paluszny, M. and R. Patterson, A family of tangent continuous cubic algebraic curves. ACM Transactions on Graphics 12 (1993), 209–232.
2. Paluszny, M. and R. Patterson, Curvas Algebraicas de Grado Tres en Modelación Geométrica. V Escuela Venezolana de Matemática (1992).
3. Paluszny, M. and R. Patterson, Geometric control of G^2-cubic A-splines. Comput. Aided Geom. Design **15** (1998), 261–287.
4. Paluszny, M., R. Patterson, and F. Tovar, G^2-composite cubic Bézier curves. Journal of Computational and Applied Mathematics 102 (1999), 49–71.
5. Salmon, G. *Higher Plane Curves*, Hodges, Foster and Figgis, Dublin, 1879.
6. Sederberg, T. Planar piecewise algebraic curves, Comput. Aided Geom. Design **1** (1984), 241–255.

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